ON SHIP WAVES ON THE SURFACE OF A VISCOUS FLUID OF INFINITE DEPTH

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PMM Vol.29, № 1, 1965, pp.186-191

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(Received September 3, 1964)

The problem of ship waves on the surface of an infinitely deep viscous fluid is considered in a simplified formulation: the ship moving with constant velocity along some curvilinear path in an unperturbed fluid is replaced by a single disturbing center in the form of a pressure pulse on a free surface (*).

1. We place the origin of the coordinate system on the free surface of the fluid in the equilibrium position and direct the z-axis vertically upward. Let the pressure pulse (ship) move toward the origin O of the coordinate system with the constant velocity c along some curvilinear course L (Fig.1).

In our subsequent discussion we largely follow the approach used in [2 and 3].

The ship's course L can be considered a curve given by the parametric equations

$$x_1 = x_1(t), \qquad y_1 = y_1(t) \tag{1.1}$$

where t is the time required for the ship to travel from some point $Q(x_1, y_1)$ to its present position O. Let us choose the zero time reference in such a way that $t = 0$ corresponds to the origin of the coordinate system. The point $Q(x_1, y_1)$ then moves along the ship's course. The shape of the waves is to be determined at the instant when the ship is at the origin. The x-axis is directed along the tangent to the ship's course L at the point O in the direction opposite to the direction of motion of the ship. The parameter t in Expressions (1.1) is the time taken with a negative sign. The vector $\mathbf{u}$ tangent to the ship's course L at the point $Q(x_1, y_1)$ is equal in absolute value to the velocity of the ship and directed opposite to the direction of the ship's motion. It is given by

$$-\mathbf{c} = \mathbf{u} = \frac{dx_1}{dt}\,\mathbf{i} + \frac{dy_1}{dt}\,\mathbf{j} \tag{1.2}$$

The point $P(x, y)$ at which the amplitude of the surface waves is computed is given by the radius vector

*) Cherkesov's paper [1] appeared after the writing of the present study; the results presented in [1], however, apply only to a rectilinear ship course.

$$\mathbf{r} = (x - x_1)\,\mathbf{i} + (y - y_1)\,\mathbf{j} \tag{1.3}$$

The angle θ in Fig.1 is the angle between the vector $\mathbf{r}$ and the velocity $\mathbf{c}$ of the ship.

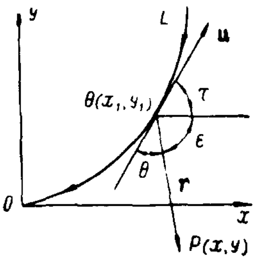

Fig. 1

The elevation of the surface $\zeta(x, y)$ at the point $P(x, y)$ is determined by integrating the elevations due to a point pressure pulse moving along L. Paper [4] contains an approximate expression for this elevation due to a point pressure pulse on the surface of a viscous incompressible fluid of infinite depth

$$\zeta_1 = -\frac{mgt^3}{8\sqrt{2}\,\pi\rho r^4}\exp\frac{-\nu g^2 t^5}{8r^4}\sin\frac{gt^2}{4r} \tag{1.4}$$

which is valid for large values of the parameters $\omega_1 = (g/\nu^2)^{1/3} r$, $\omega_2 = gt^2/(4r)$. Since $\lambda = (g/\nu^2)^{1/3}$ is large in the case of water, Formula (1.4) is also valid for small (non-zero) values of r (for $\nu = 0$, (1.4) becomes the familiar expression for an ideal fluid) [2 and 3]. In Formula (1.4), t is the time elapsed since the instant of application of the pulse, and r is the distance from the point of application of the pulse to the given point on the surface of the fluid.

The integral effect for the motion of a point pulse may be written as

$$\zeta(x, y) = -\frac{mg}{8\sqrt{2}\,\pi\rho}\int_0^\infty \frac{t^3}{r^4}\exp.\frac{-\nu g^2 t^5}{8r^4}\sin\frac{gt^2}{4r}\,dt = \operatorname{Im}\int_0^\infty \psi(t)\,e^{i\varphi(t)}\,dt \tag{1.5}$$

$$\psi(t) = -\frac{mg}{8\sqrt{2}\,\pi\rho}\,\frac{t^3}{r^4}\exp\frac{-\nu g^2 t^5}{8r^4}, \qquad \varphi = \frac{gt^2}{4r} \tag{1.6}$$

2. Let us apply the stationary phase method to the computation of integral (1.5). (As the large parameter we can take the dimensionless quantity $gR/(4c^2)$, where R is the distance from the ship to the point on the surface of the fluid under consideration).

The phase stationarity condition $d\varphi/dt = 0$ leads to the relation

$$\frac{dr}{dt} = \frac{2r}{t} \tag{2.1}$$

From condition (1.3) we find

$$r\,dr/dt = -\left[(x - x_1)\,dx_1/dt + (y - y_1)\,dy_1/dt\right] = cr\cos\theta \tag{2.2}$$

Making use of Expressions (2.1) and (2.2) we obtain the phase stationarity condition

$$r = {}^1/_2\,ct\cos\theta \tag{2.3}$$

Condition (2.3) gives us those points Q_1 which participate in the formation

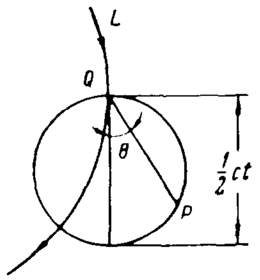

Fig. 2

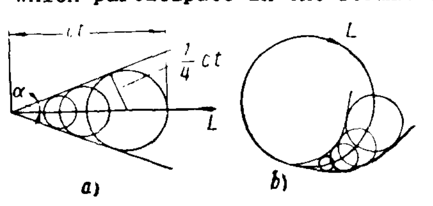

Fig. 3

(within the limits of our approximation) of the disturbance at the point P.

The effect of other points is cancelled by mutual interference[1]. The points Q_1 thus determined are called effect points.

Expression (2.3) is the equation of a circle in polar coordinates with the pole at the point Q lying on the ship's course L. Thus the point Q is the effect point for all points of the circle, whose diameter is the tangent to the curve of the ship's course at the point Q lying at one end of this diameter (Fig.2). The disturbance produced by the ship does not act on the entire surface of the fluid, but only on the points that lie on the effect circles of all points Q along the ship's course. The surface waves produced by the moving ship are limited to the region covered by the effect circles, i.e. to the region bounded by the envelope of this one-parameter family of curves. We arrive at results familiar to us from the theory of ship waves on the surface of an ideal fluid.

a) The contours of the disturbance region for any course of a ship moving with constant velocity are found, as in the case of an ideal fluid, by constructing the envelope of the effect circles for points lying along the ship's course. Fig.3 shows cases where the ship moves with a constant velocity c along a straight path and in a circle. In the case of a straight course, the envelope turns out to be a pair of straight lines, and the disturbance is confined to a sector of angle 2α, where

$$\alpha = \sin^{-1} {}^1/_3 = 19°28' \tag{2.4}$$

Strictly speaking, the disturbance produced by the ship will not be equal to zero outside this sector, but it will be of an order of magnitude lower than the disturbance within that region.

b) The effect points Q_1 corresponding to a given point P in the case of motion along a straight course with constant velocity c may be found by the usual method (Fig.4): the point P is connected by a straight line with the point O; the segment OP is bisected at the point C; a circle is then constructed with CP as its diameter and the points of intersection M_1 and M_2 of this circle with the Ox-axis are found. Next, laying out the segments $M_1Q_1 = OM_1$ and $M_2Q_2 = OM_2$; one obtains the required positions Q_1 and Q_2 of the effect points. Depending on the position of the point P, it is possible to have two, one, or no points of intersection M_1 and M_2.

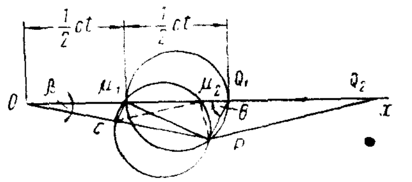

Fig. 4

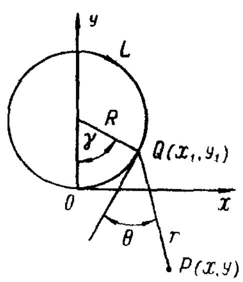

Fig. 5

3. The analytic investigation of constant-phase curves is carried out in the same way as for an ideal fluid. It is convenient to express phase stationarity condition (2.3) by introducing the quantity a that has the dimension of length

$$a = 2c^2 \varphi/g = {}^1/_2 c^2t^2/r \tag{3.1}$$

We find the phase stationarity conditions from (2.3) and (3.1)

$$ct = a\cos\theta, \qquad r = {}^1/_2 a\cos^2\theta \tag{3.2}$$

Let us determine the constant phase curves for circular and straight ship courses assuming that the vessel travels with a constant velocity. The former case was considered for an ideal fluid by Sretenskii [5]. The straight course case can be considered as a limiting case of the circular course

(Fig.5) for the ship's position $Q(x_1, y_1)$ at a previous instant we have

$$x_1 = R \sin\gamma, \quad y_1 = R(1 - \cos\gamma), \quad \gamma = ct / R \tag{3.3}$$

where R is the radius of the ship's circular course, t is the time required by the ship to move from Q to O, and c is the constant velocity of the ship. The coordinates of the point P at which the disturbance is sought are given by the expression

$$x = x_1 - r\cos(\gamma + \theta), \qquad y = y_1 - r\sin(\gamma + \theta) \tag{3.4}$$

Replacing x_1 and y_1 by their expressions in (3.3) and making use of the relation for r in (3.2), we obtain

$$x = R\sin\gamma - {}^1/_2 a\cos^2\theta\cos(\gamma+\theta), \qquad y = R(1-\cos\gamma) - {}^1/_2 a\cos^2\theta\sin(\gamma+\theta) \tag{3.5}$$

Let us find the geometric locus of the points for which the phase φ, i.e. the quantity a in Formula (3.1) is constant. It is convenient to introduce the dimensionless parameter

$$\varkappa = a / R \tag{3.6}$$

Applying the first relation of (3.2), we find that

$$\gamma = \frac{ct}{R} = \frac{a}{R}\cos\theta = \varkappa\cos\theta \tag{3.7}$$

Relations (3.5) can then be written out in dimensionless form

$$\begin{aligned} x / R &= \sin(\varkappa\cos\theta) - {}^1/_2\varkappa\cos^2\theta\cos(\theta + \varkappa\cos\theta) \\ y / R &= 1 - \cos(\varkappa\cos\theta) - {}^1/_2\varkappa\cos^2\theta\sin(\theta + \varkappa\cos\theta) \end{aligned} \tag{3.8}$$

These equations express the constant-phase curves in terms of the parameter θ. Each value of $\varkappa$ gives one such curve, since specification of $\varkappa$ (for a fixed value of the circular course radius R) is equivalent to specification of the phase φ. Fig.6 shows several constant phase curves and the contours of the disturbance region computed on the basis of Equations (3.8). We obtain two systems of waves called the system of divergent and the system of transverse waves (Fig.6).

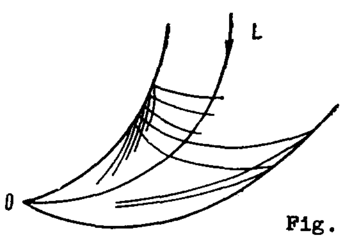

Fig. 6

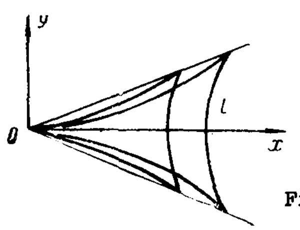

Fig. 7

Setting $R \to \infty$ and $\varkappa \to 0$ in Equations (3.8) and stipulating that $R\varkappa = a$ by virtue of (3.6), we arrive at the equations of the constant-phase curves for the case of a straight ship course

$$x = {}^1/_2 a(2 - \cos^2\theta)\cos\theta = {}^1/_2 a(1 + \sin^2\theta)\cos\theta, \qquad y = {}^1/_2 a\cos^2\theta\sin\theta \tag{3.9}$$

Fig.7 shows the results of computations carried out on the basis of Formulas (3-9). Comparison with photographs of waves produced by a moving ship found in [2] indicates good agreement.

From Equations (3.9) we have

$$\frac{dx}{dt} = -\frac{a}{2}(3\sin^2\theta - 1)\sin\theta, \qquad \frac{dy}{dt} = \frac{a}{2}(3\sin^2\theta - 1)\cos\theta \tag{3.10}$$

This implies that $dy/dx = -1/\tan\theta$, which in turn means (Fig.8) that the constant-phase lines are orthogonal to the lines extended to the effect points. The values $\theta = \theta^*$ for which $3\sin^2\theta - 1 = 0$ have special corresponding points on the curves at the disturbance boundary. The effect points Q_1 and Q_2 coincide for these boundary points. Clearly, these points are apical, i.e. first-order cusps. It is also clear that the system of divergent waves is produced (for $y < 0$) as θ varies over the interval $\theta^* \leqslant \theta \leqslant \pi/2$, while the transverse waves correspond to values of θ in the interval $0 \leqslant \theta \leqslant \theta^*$. For $y > 0$, the angle θ assumes negative values accordingly.

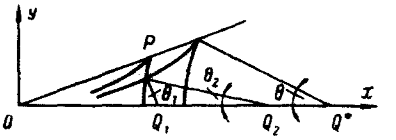

Fig. 8

In addition, it is easy to see that to any point along the ship's course ($\theta = 0$) there corresponds one and only one effect point of the type Q_2 and that this point does not coincide with P, since the divergent waves encounter the ship's course only at the point $O(x = 0, y = 0)$. For this reason, the stationary phase method is also applicable for computing the amplitudes of waves for points that lie on the ship's course

Formulas (3.9) for $\theta = 0$ give us

$$OB = {}^1/_2\, a = c^2\varphi / g = c^2\delta / g \qquad (\varphi = \delta = \text{const}) \tag{3.11}$$

Hence, the length of the transverse waves is

$$\lambda = 2\pi c^2 / g \tag{3.12}$$

their speed of propagation is $(g\lambda / 2\pi)^{1/2} = c$, i.e. the velocity of the ship.

4. Let us consider the amplitude of surface waves given by our approximation. To do this, we compute φ, $d^2\varphi/dt^2$, and $d^3\varphi/dt^3$ for values of t that satisfy the phase stationarity condition $d\varphi/dt = 0$. Taking into account (2.2), it is easy to see from (2.1) that

$$\frac{d^2\varphi}{dt^2} = \frac{g}{2r}\left(1 - \frac{t^2}{2r}\frac{d^2r}{dt^2}\right) \tag{4.1}$$

The value of $d^3\varphi/dt^3$ at points where $d^2\varphi/dt^2 = 0$ is given by Equation

$$\frac{d^3\varphi}{dt^3} = -\frac{gt^2}{4r^2}\frac{d^3r}{dt^3} \tag{4.2}$$

Let us express our results in terms of the parameter θ instead of the variable t. From Formula (2.4) we have $dr/dt = c\cos\theta$, so that

$$\frac{d^2r}{dt^2} = -c\sin\theta\frac{d\theta}{dt} \tag{4.3}$$

where c is the constant speed of the ship. To compute $d\theta/dt$, we introduce the angles ε and τ shown in Fig.1. Then

$$\theta = \pi - (\varepsilon + \tau), \qquad \frac{d\theta}{dt} = \left(-\frac{d\varepsilon}{ds} - \frac{d\tau}{ds}\right)\frac{ds}{dt} = -c\left(\frac{d\varepsilon}{ds} + \frac{d\tau}{ds}\right) \tag{4.4}$$

where s is the arc length along the curve L. Further, we have $d\tau/ds = 1/R$, where R is the radius of curvature of the curve L,

$$\varepsilon = \tan^{-1}\frac{y - y_1}{x - x_1}, \qquad \frac{d\varepsilon}{ds} = -\frac{1}{r^2}\left[(x - x_1)\frac{dy_1}{dt} - (y - y_1)\frac{dx_1}{dt}\right] = \frac{\sin\theta}{r} \tag{4.5}$$

Applying Formulas (4.3) to (4.5) and (3.2), we find that

$$\frac{d^2\varphi}{dt^2} = \frac{g}{2r}\left(1 - 2\tan^2\theta - \frac{a}{R}\sin\theta\right) = \frac{g}{2r}\,\frac{1 - 3\sin^2\theta - (q/R)\sin\theta(1 - \sin^2\theta)}{\cos^2\theta} \tag{4.6}$$

where a is given by Formula (3.1).

From relation (4.6) we may conclude that the only courses for which the wave wake feft by the ship experiences displacement unchanged like a solid body are the straight and circular courses, i.e. those for which $R =$ const .

Finally, let us determine the amplitude $\zeta(x, y)$ of the waves by the stationary phase method. The contribution of stationary phase point t_0 to integral (1.5) is given by Expression

$$\zeta(x, y) \sim \mathrm{Im}\left\{\psi(r, \theta)\left(\frac{2\pi}{|\varphi''(r, \theta)|}\right)^{1/2}\exp\left[i\left(\varphi(r, \theta) \pm \frac{\pi}{4}\right)\right]\right\} \qquad (\varphi''(r, \theta) \neq 0) \quad (4.7)$$

where r and θ are polar coordinates that define the position of the stationary phase point on the course L relative to the point (x, y) (Fig.1). The sign in front of the term $\pm \frac{1}{4}\pi$ in the exponent is the same as that of $\varphi'' = d^2\varphi/dt^2$.

We shall confine ourselves to a consideration of the amplitudes for the straight course of the ship only. From (4.6) we have

$$\frac{d^2\varphi}{dt^2} = \frac{g}{2r}\frac{1 - 3\sin^2\theta}{\cos^2\theta} \quad (4.8)$$

As was shown earlier, each point in the disturbance region has two corresponding values of θ that define the effect points; let us call them θ_1 and θ_2 . One of these belongs to the system of transverse waves $(0 \leqslant \theta_1 < \theta^* = \sin^{-1} 1/\sqrt{3})$, and the other – to the system of divergent waves $(\theta^* < \theta_2 \leqslant {}^1/_2\pi)$. In the first instance the derivative $d^2\varphi/dt^2$ is positive, and in the second – negative. Formula (4.7) is inapplicable on the boundary of the disturbance region, where $\varphi'' = 0$. We consider this case separately.

For points within the disturbance region $(0 \leqslant \theta_1 < \theta^*,\ \theta^* < \theta_2 \leqslant {}^1/_2\pi)$ we obtain

$$\zeta(x, y) \sim \mathrm{Im}\left\{\psi(r_1, \theta_1)\frac{\sqrt{2\pi}}{\sqrt{|\varphi''(r_1\theta_1)|}}\exp\left[i\left(\varphi(r_1, \theta_1) + \frac{\pi}{4}\right)\right] + \right.$$
$$\left. + \psi(r_2, \theta_2)\frac{\sqrt{2\pi}}{\sqrt{|\varphi''(r_2, \theta_2)|}}\exp\left[i\left(\chi(r_2, \theta_2) - \frac{\pi}{4}\right)\right]\right\} \quad (4.9)$$

Taking into account Formulas (2.3), (3.1) and (1.6), we can rewrite Formula (4.9) as

$$\zeta(x, y) \sim -\frac{2m\sqrt{g}}{\rho c^3\sqrt{\pi}}\left\{\exp\left(-\frac{2\nu g^2 a_1 \sec^3\theta_1}{c^5}\right)\frac{\sec^3\theta_1}{a_1^{1/2}\sqrt{|1 - 3\sin^2\theta_1|}}\sin\left(\frac{ga_1}{2c^2} + {}^{\pi}/_4\right) + \right.$$
$$\left. + \exp\left(-\frac{2\nu g^2 a_2 \sec^3\theta_2}{c^5}\right)\frac{\sec^3\theta_2}{a_2^{1/2}\sqrt{|1 - 3\sin^2\theta_2|}}\sin\left(\frac{ga_2}{2c^2} - \frac{\pi}{4}\right)\right\} \quad (4.10)$$

The two wave systems are therefore shifted in phase by $\pi/2$ at every point where $a_1 = a_2$ (on the disturbance region boundary). Hence, if we were to plot the systems of divergent and transverse waves in Figs. 6 and 7 making due allowances for this difference in phase at the boundaries, the agreement of the theoretical results with photographs of waves produced by the motion of a ship would be seen to be even better.

The wave amplitudes decay as $a_i^{-1/2}\exp(-2\nu g^2 c^{-5} a_i \sec^3\theta_i)$. In contrast to an ideal fluid, the amplitudes of the divergent waves at the origin O do not become infinite. For $\nu = 0$, (4.10) gives us the formula for an ideal fluid.

Formula (4.10) implies that the amplitudes of both wave systems become infinitely large for $\theta = \theta^*$, i.e. for points on the boundary of the disturbance region. But asymptotic formula (4.7) is not applicable at such points, since there $\varphi'' = 0$.

To determine the amplitude of waves along the boundary of the region $(\theta = \theta^*)$, let us make use of the formula [2]

$$\zeta(x, y) \sim \mathrm{Im}\left\{\frac{\Gamma(^1/_3)}{\sqrt{3}}\psi(\theta^*)\left(\frac{6}{|\varphi'''(\theta^*)|}\right)^{1/3} e^{i\varphi(\theta^*)}\right\} \quad (4.11)$$

Replacing the derivative $d\theta/dt$ by $(1/r)_c \sin\theta$ in (4.3) and differentiating the result with respect to t, we have

$$\frac{d^3r}{dt^3} = -\frac{c^3\cos\theta\sin^2\theta}{r^2}$$

Recalling that $r = {}^1/_2 c\cos\theta = {}^1/_2 a\cos^2\theta$, we find that for $\theta = \theta^*$ Formula (4.2) gives us

$$\frac{d^3\varphi}{dt^3} = \frac{4gc}{a^2}\frac{\sin^2\theta^*}{\cos^5\theta^*} = \frac{3gc}{a^2}\frac{\sqrt{6}}{2} \tag{4.12}$$

Substituting the resulting expressions in Formula (4.11), we finally obtain

$$\zeta(x, y) \sim -\frac{9mg^{2/3}\Gamma({}^1/_3)}{2\pi\rho\, 3^{2/3} c^{10/3} a^{1/3}} \exp\left(\frac{-\nu g^2\, 3\sqrt{6}\, a}{2c^5}\right)\sin\frac{ga}{2c^2} \tag{4.13}$$

The wave amplitudes now decay as $a^{-1/3}\exp[-{}^1/_2\nu g^2 c^{-5}\, 3\sqrt{6}\, a]$. For $\nu = 0$ Formula (4.13) becomes the corresponding formula for an ideal fluid.

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Translated by A.Y.

Editorial Note :

*) Stoker, J.J., Water Waves: the Mathematical Theory with Applications. Interscience Publishers, Inc., New York, 1957.